The Voronoi Projection

Philippe Rivière*

Visionscarto, LIRIS

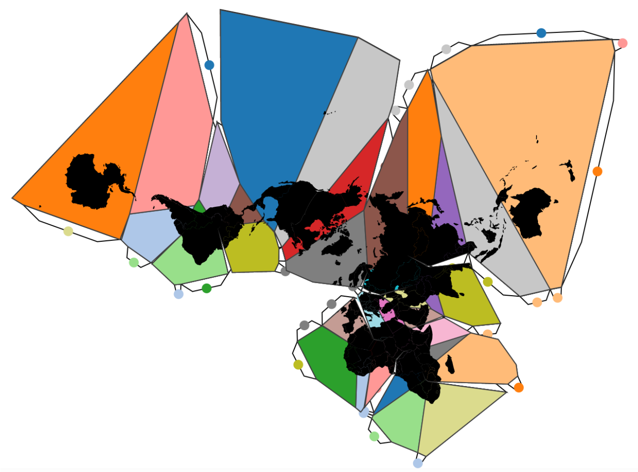

Figure 1: Voronoi projection with 38 faces centered on the 38 largest countries' centroids – and color-coded tabs

ABSTRACT

In this poster we introduce the Voronoi projection, a novel construction method for polyhedral cartographic projections.

The system allows the creation of maps that follow explicit design goals, encoded in the choice of a set of centers connected by an arbitrary spanning tree.

From centers following a regular arrangement of points, this construction method can recreate the traditional polyhedral projections such as a cube or dodecahedron. However, as it does not depend on the rigid symmetries of the platonic solids, it continues to offer a solution when these centers are displaced by perturbations or by the addition of more centers.

Furthermore, these centers and tree can be derived from arbitrary rules, which we demonstrate by specifying a cost function that prevents the fragmentation of land – alternatively, ocean.

Having specified these rules, we can push the number of faces higher, and observe the emergence of specific shapes of the world, thus giving a visual representation of a geographic computation.

We offer a software package that builds these projections and integrates into the D3.js library, allowing in-browser and offline rendering of vector features and raster images, as well as interactivity. Finally, we provide distortion analysis.

An interactive demonstration is available online, see [5].

Index Terms: Maps—Projections—Voronoi

1 INTRODUCTION

Cartographic projections map the Earth onto flat screen or paper. An unavoidable consequence of the topological differences between these surfaces is that a projection cannot at the same time preserve areas (equivalence), angles (conformality), and avoid cuts (continuity). This fundamental challenge has been the main driver of invention of map projections over the course of 2,000 years, mostly focused on finding reasonable and useful compromises between the two first properties, and giving birth to hundreds of projections [6].

A handful of cartographers have introduced cuts, in the form of polyhedral projections with regular arrangements of a limited number of faces – from Leonardo Da Vinci's octaedron, refined by B.J.S. Cahill and Gene Keyes, or Buckminster Fuller and the iconic Dymaxion™ projection [4].

One inventor (Jarke J. van Wijk) has proposed to abandon the global continuity requirement, by introducing "myriahedral projections" – with unlimited cuts and a general technique to patch up large numbers of faces [7].

The Voronoi projection is a similar proposal, but following a novel and distinct mathematical route.

The set of faces and the spanning tree can be built around explicit design goals. One example is to see how simple polyhedral projections evolve if the positions of their faces are randomly perturbed, or if new faces appear on what used to be corners (figure 2), thus giving a visual and intuitive feedback on what polyhedral projections "do" to the spherical geometries, in terms of distortions and cuts.

Another example is to connect random centers through land rather than ocean (see figures 1 and 3) – or vice-versa (figure 4).

2 GENERAL METHOD TO BUILD VORONOI PROJECTIONS

The Voronoi projection can be built through the following steps:

A) Define a set a of n sites on the surface of the sphere. These sites can be geographically meaningful (for example, the largest airports on Earth, or centroids of specific regions of interest), regularly spaced or randomly dispersed.[1]

B) Compute the (spherical) Delaunay Triangulation / Voronoi Diagram of this set of sites.

C) Define a cost for each connection between sites. The cost function is arbitrary. It may be based on the geodesic distance between the sites, multiplied by a factor that represents our preferences, e.g. for land-based or ocean-based graphs.

D) Compute the spanning sub-graph that minimizes the total cost, i.e. the minimal spanning tree (MST) of the full Delaunay graph.

E) Select a starting site C_0, and project its Voronoi cell with a gnomonic projection centered on C_0. This gives a well-defined face transformation; the gnomonic projection ensures that the Voronoi cell, a spherical polygon, is projected to a planar polygon. And, by property of the gnomonic projection, each edge of the spherical face is projected to a straight line, allowing perfect stitching of the faces along the tree.

F) Recursively follow the spanning tree. There is a unique rotation & translation transform that attaches the face's projected image to its parent through their shared edge.

*e-mail: philippe.riviere@visionscarto.net

[1] A requirement is that the union of the n hemispheres centered on each of these sites must cover the whole sphere — otherwise the projection will not fully cover the sphere, as the gnomonic projection used in step E has a maximum range of 90°.

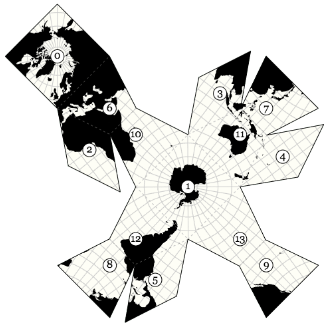

Figure 2: A cube, augmented by 8 faces.

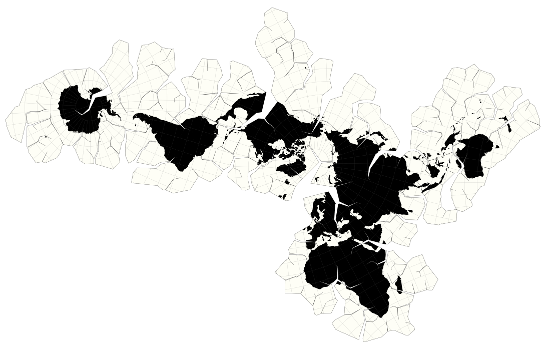

Figure 3: Voronoi projection with 400 randomly distributed faces – with links that favor in-land connections

3 Distortion analysis

The scale of the projection uniform can be shown, by induction, to be the same at all sites. On each face, it is an increasing function of the angular distance from the center (ρ). The maximum distortion thus happens on the face with the largest radius. The scale at that point equals $\sec^2 \rho$. Given n faces with approximately the same size, this can be shown to be approximately $1 + 4\pi/n$ for a large value of n. In other words, for 100,000 faces, the maximum scale distortion is of the order of 10^{-4}.

4 Tools

The construction of the Voronoi projection draws heavily on the D3.js ecosytem [1]. To compute the spherical Voronoi diagram, we use Loren Petrich's algorithm [3].

To compute the MST, a simple Kruskal algorithm is implemented; the cost function of a connection (A;B) is defined as the geodesic distance between A and B, times a factor that depends on A's and B's distances to shore. Distance to shore is a way to see if a point is "deep in land" or "far in the ocean". We approximate this by computing the minimum number of hops on the Delaunay graph to reach a connection between a land point and a water point. (This heuristic contrasts a lot with van Wijk's approach.)

To compute the polyhedral projection and actually use it to project GeoJSON vector features onto the screen, we use Jason Davies' clipping algorithm, implemented in d3-geo-polygon [2], the D3.js module dealing with polyhedral projections.

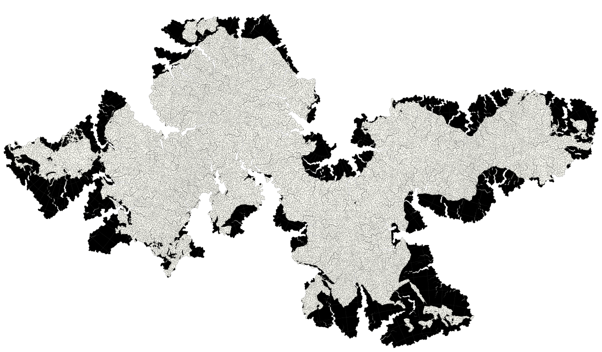

Figure 4: Ocean-centered Voronoi projection with 100,000 faces

5 Computation time

The complexity of the algorithms employed to build the projection is at worst $O(n^2)$. In practice, our implementation solves the 100,000 faces projection in 36 minutes on a regular laptop [5].

The inverse projection, which allows for interaction, as well as projection of rasters such as satellite images, can be computed with $O(n^{1/2})$ operations per pixel, walking the Voronoi diagram to search for the closest face.

6 Further research

In some cases a branch of the MST might be projected on the plane in such a way that it overlaps another branch, making it impossible to print. We currently have no means to avoid this during the calculation of the MST (though we can detect the situation *ex post* and eliminate the offending projection).

Although our 12-year old paper toy specialist Ines Rivière-Poupon has empirically demonstrated that the 38 faces projection (fig. 1) can be folded back to a solid model, this remains to be proven mathematically.

7 Conclusion

We have introduced the construction of a novel cartographic projection, based on the Voronoi Diagram of an arbitrary set of sites on the sphere, and an equally arbitrary spanning tree. We reckon that this projection's usefulness in creating operable maps is still to be demonstrated, but we have shown it has interesting properties:

- for fun and art, to create quirky paper toys representing the Earth and any other spherical body;
- as a way to visualize a spherical minimal spanning tree as the solution of a constraint system.

Most of all, these images can be used as a pedagogical tool as they allow – as van Wijk puts it for the myriahedral projections – to "visualise, just like the Tissot indicatrix, the distortion that occurs when a non-interrupted map is used, and can be used to explain the basic problem of map projection".

Acknowledgments

The author wishes to thank Enrico Spinielli, Aurélien Tabard and Romain Vuillemot.

References

[1] M. Bostock, V. Ogievetsky, and J. Heer. D³ data-driven documents. 17(12):2301–2309. doi: 10.1109/TVCG.2011.185

[2] J. Davies, M. Bostock, and P. Rivière. d3-geo-polygon – https://github.com/d3/d3-geo-polygon.

[3] L. Petrich. d3-geo-voronoi – https://github.com/d3/d3-geo-voronoi.

[4] P. Pedzich. Image of the world on polyhedral maps and globes. 48(4). doi: 10.1515/pcr-2016-0014

[5] P. Rivière. The voronoi projection – https://beta.observablehq.com/@fil/voronoi-projection.

[6] J. P. Snyder. *Flattening the earth: two thousand years of map projections*. The University of Chicago Press. OCLC: 26764604.

[7] J. J. van Wijk. Unfolding the earth: Myriahedral projections. 45(1):32–42. doi: 10.1179/000870408X276594